MECHANIZED MATHEMATICS AND ITS APPLICATIONS, VOL. 6, NO. 1, NOVEMBER 2007, 33:40

A Proof of the Jordan Curve Theorem via the Brouwer Fixed Point Theorem

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Abstract – The aim of the paper is to report on MIZAR codification of the Jordan curve theorem, a theorem chosen as a challenge to be completely verified using formal methods at the time when they started being commonly used. Formalization was done based on proofs taken from the literature, where theorems mentioned in the title of the paper from "Brouwer's Fixed Point Theorem and the Jordan Curve Theorem" by David Gauld and "Algebraic Topology" by Allen Hatcher, respectively. The entire work was done through a strong cooperation between Polish and Japanese universities, especially Białystok University and Shinshu University, Nagano.

Keywords – homotopy theory, Jordan's curve theorem, Brouwer's fixed point theorem.

1. Introduction

In the early 1990s, when the general use of formal methods started to become common, committees using provers and proof checkers set out to fix some tasks to test the existing systems. From among the many choices, the committees decided on a full formalization of a chosen nontrivial mathematical theorem and the choice for the challenge was the Jordan curve theorem .

This theorem was formulated by Camille Jordan in his textbook "Cours d'Analyze de l'École Polytechnique" in 1887. From that time many different formulations of the theorem have arisen, but all are equivalent to the statement: *any simple closed curve in the plane separates the plane into two disjoint regions, the inside and the outside.*

Since Jordan's times many different proofs of the theorem have appeared. Some of them were false, some had small gaps. Unfortunately, the public opinion is that even the original proof given by Jordan was false, despite a defense of the proof by Hales in [9]. The same opinion also contends that the first correct proof of the Jordan curve theorem was given by Oswald Vablen in 1905. Different proofs of the theorem use different mathematical backgrounds, intermediate lemmas, and techniques like the theory of planar graphs or the Brouwer fixed point theorem to reach the target.

The technique of internal and external approximations of a given curve was firstly chosen to be completely verified by a computer using the MIZAR system. The proof is described in [23] and its formal verification started in 1991 with the work in [4]. Soon after in 1992, a very special case of the theorem stating that a rectangle satisfies the Jordan property was completed

Manuscript received August 22, 2007; revised October 9, 2007.

([20]). The next part, still a special case, involved curves composed from vertical and horizontal segments only and was finished in 1999 ([25, 11]). Unfortunately, the general case about simple closed curves following [23] was never verified by MIZAR but a number of milestone steps were reached, namely: the existence and uniqueness of a nonempty external (outside, unbounded) component (called `UBD` in MIZAR terms) and the existence of a nonempty internal (inside, bounded) component (called `BDD` in MIZAR terms) of the complement of a given curve in [21] and [24], respectively.

When the last part, that is the uniqueness of a bounded component, remained to be proved, Shidama and Korniłowicz completed the mechanization of the Brouwer fixed point theorem in MIZAR . They then decided to give up following Takeuchi and Nakamura's script ([23]) and instead complete the proof of the Jordan curve theorem using the Brouwer fixed point theorem following [5]. The final work was finished in 2005 and is stored in [16] – (relevant files are available from the MIZAR home page, `http://mizar.uwb.edu.pl`). Since the MIZAR formalization is almost a copy of the paper proofs, with of course proving of all simple auxiliary proof steps mentioned but not proofs in [5], we do not explain all proofs here. In the following the reader is directed to the source paper as necessary.

Here, we can say that the year 2005 was, in some sense, decisive in the history of proofs of the theorem. Two different formalizations, entirely verified by computers, were completed that year. Separately from the work mentioned above, Thomas C. Hales completed his formalization in HOL first in January. All files containing his formalization are available at his home page `http://www.math.pitt.edu/~thales/`.

Moreover, the year 2005 was the 100th anniversary of the first correct proof of the Jordan curve theorem given by Oswald Vablen.

The aim of this paper is not to present the whole proof of the theorem, but to mark crucial points in the proof, since work done by the author of the article is not original from a mathematical point of view, even if he has found some small gaps in proofs taken from the literature on which the paper is based. The role of the paper is underlying the fact that practical formalization of non trivial mathematics is possible, even more, it is continuously being done using different proof assistants - proof checkers and provers. Moreover, detailed proofs of all of the lemmas and theorems used in the proof of the Jordan curve theorem can be found in the MIZAR articles listed in the references.

Section 2. describes in short a proof of the Brouwer fixed point theorem. The proof is based on two main facts: 1) no circle is a retract of a disk and 2) a fundamental group of a circle is isomorphic to integers (in fact, non triviality of the group is enough to prove the Brouwer fixed point theorem) ([15]). Assuming, on the contrary, that the theorem is false, we show that a circle is a retract of a disk. Next, using the retraction we prove that any loop in the circle is homotopic to the constant loop which contradicts with the infiniteness of the fundamental group of a circle.

Section 3. describes briefly the proof of the Jordan curve theorem. But, to understand, or simply to find out all of the details of the proof, the reader is strongly invited to read the original paper by David Gauld ([5]) or the related MIZAR files.

2. Brouwer's fixed point theorem

The Brouwer fixed point theorem is well-known in the theory of fix points. It was stated by L. E. J. Brouwer in 1909, but proved first by J. Hadamard in 1910 and then by Brouwer in 1912 using a different approach. The theorem is usually stated as: *"any continuous function from a closed unit ball in n-dimensional Euclidean space into itself has a fix point"*, but can be easily extended to any homeomorphic image of a closed unit ball.

The theorem is considered an achievement of algebraic topology for its elegant proofs using machinery served by the theory, like homology groups for $n \geq 3$ and fundamental groups in the case of $n = 2$. If $n = 1$, the theorem can be proved directly from the intermediate value theorem. Other proofs can be obtained as equivalences to, for example, Sperner's lemma or the Hex game.

In next two subsections we present brief outlines of proofs for 1- and 2-dimensional cases encoded in MIZAR and stored in [26] and [18], respectively.

2.1 1-dimensional case

Because unit disks on the real line are intervals $[-1,1]$ we can formulate the theorem as: every continuous function $f : [-1,1] \mapsto [-1,1]$ has a fixed point, that is there exists a point $-1 \leq x \leq 1$ such that $f(x) = x$.

To prove the theorem let us consider the following cases:

- if $f(1) = 1$ or $f(-1) = -1$, it is obvious that 1 and -1 satisfy the thesis, respectively.
- when $-1 < f(-1)$ and $f(1) < 1$ is a consequence of the "intermediate value theorem" which states: for every real map F defined on the closed interval $[a,b]$, for all reals x_1, x_2 such that $a \leq x_1 < x_2 \leq b$ and for every y satisfying $F(x_1) < y < F(x_2)$ or $F(x_2) < y < F(x_1)$ there exists a real x satisfying $x_1 < x < x_2$ and $F(x) = y$. If we define function $F : [-1,1] \mapsto \mathbb{R}$ as $F(x) = f(x) - x$, $a = x_1 = -1$, $b = x_2 = 1$ and $y = 0$, they satisfy assumptions of the intermediate value theorem $F(1) < 0$ (since $F(1) = f(1) - 1$ and $f(1) < 1$) and $0 < F(-1)$ (since $F(-1) = f(-1) - -1$ and $-1 < f(-1)$) and then there exists an x such that $F(x) = 0$, that is $f(x) = x$. The x satisfies the conditions of the Brouwer fixed point theorem.

2.2 2-dimensional case

In MIZAR language, the 2-dimensional case of the Brouwer fixed point theorem can be formulated as:

```
theorem :: BROUWER:14
for r being non negative (real number),
    o being Point of TOP-REAL 2,
    f being continuous Function of Tdisk(o,r), Tdisk(o,r) holds
f has_a_fixpoint;
```

As we mentioned above there are a number of different proofs of the theorem using completely different approaches. To codify the theorem in MIZAR we chose a proof based on algebraic topology which is nicely presented in [10]. Unfortunately, when we started the formalization of the proof, the Mizar Mathematical Library (MML) was not prepared well for the work in the sense that there were not many articles on algebraic topology stored in the MML (in fact there was only one, [6]). We then first started the codification work from the development of general theory required to define fundamental groups and proofs of its properties for simple closed curves with the fundamental one concerning isomorphism to integers, see [8] for details. Having all this prepared, we verified the following proof.

Let us assume that a given map f has no fixed points. Since $f(x) <> x$ for every x of the disk, one can define a function r from the disk into the circle by putting $r(x)$ as the intersection point of the circle with a half-line passing from $f(x)$ through x. Because $f(x) <> x$, it is possible to express r using the formula counting the line passing through two different points on the plane which is useful in a proof of continuity of r. Moreover, r is the identity on the circle and then r is a retraction of the disk onto the circle which is impossible.

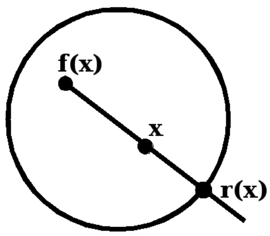


Fig. 1. Construction of the retraction r

To prove that a circle in not a retract of a disk, let us consider a loop f_0 at a given point x_0 of the circle. Since a disk is convex and all loops in a convex subspace of the Euclidean space are homotopic, one can consider a homotopy F between f_0 and a constant loop at x_0 in the disk. But then $r * F$ is a homotopy between $r * f_0$ and the constant loop in the circle. Since r is the identity on the circle $r * f_0 = f_0$ and the fundamental group of the circle is trivial, that is a contradiction.

3. Jordan's curve theorem

3.1 Formulation

As we mentioned in Section 1. there are different formulations of the Jordan curve theorem, but all state that any simple closed curve disjoins the plane into two parts – bounded and unbounded – and is their common border. The statement which we verified is formulated as:

```
theorem :: JORDAN:99
  for C being Simple_closed_curve holds C is Jordan;
```

where, the property `Jordan` is defined as:

```
definition
```

```
  let S be Subset of TOP-REAL 2;
  attr S is Jordan means
:: JORDAN1:def 2
  S` <> {} &
  ex A1, A2 being Subset of TOP-REAL 2 st
    S` = A1 \/ A2 & A1 misses A2 & (Cl A1) \ A1 = (Cl A2) \ A2 &
    for C1, C2 being Subset of (TOP-REAL 2)|S` st C1 = A1 & C2 = A2 holds
        C1 is_a_component_of (TOP-REAL 2)|S` &
        C2 is_a_component_of (TOP-REAL 2)|S`;
end;
```

Observe that our formulation does not take into account boundness excplicitely. It only claims the existence of two disjoint components of the complement of a given curve, which give the entire complement as their union. If they exist, one must be bounded and one unbounded.

3.2 Proof

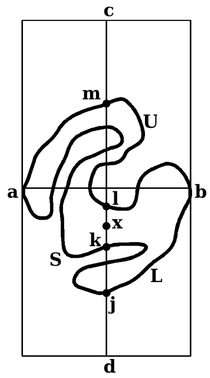
To proof the theorem two particular sets must be constructed. The idea is two find two different points out of the given curve and to show that they 'generate' required sets – unique components of the complement. Since simple closed curves are compact, and then bounded, it is easy to find a point laying in the unbounded part of the complement and then to proof the unbounded case, what was done in [21].

The difficult part of the whole proof of the Jordan curve theorem is finding a unique component of the complement of a given closed curve, which is bounded. The question is how to construct a point lying inside a bounded component of the complement of the curve, that is the existence of a bounded component. Having this point, the last part of the proof is to show that every bounded component of the complement must contain the point, that is the uniqueness of the bounded component.

The very elegant construction of the point is presented in Figure 2, which is repaint from [5], where S is a given curve, U is an arc going from a to b through m, and L is the rest part of S. Observe, that the construction is done not for the original curve S, but for its affinical image closed in a fixed rectangle (determined by points a, b, c, d) what allows to omit some difficulties with location of the original curve on the plane. The meaning of all points is clear from the picture.

We then decided to formalize the proof presented in [5], but with two exceptions. The first exception is trivial since we decided not to close the transformed curve inside rectangle with height 4, as in the paper, but with height 6 what made certain calculations easier. The second exception is more serious: the proof of Lemma 5.6 of [5] assumes existence of a nonempty component of $\mathbb{R}^2 \setminus S$ what is not excplicitely mentioned in the statement of the lemma. We then added the assumtion to the lemma. Fortunately, this did not lead to any critical situation and Lemma 5.6 could be still used in the proof of the main theorem, since it was refered to in the part of the proof, where uniqueness of the bounded component was being proven and its existence was already known.

So, the logical structure of the whole proof is:



$a = (-1,0)$, $b = (1,0)$
$c = (0,2)$, $d = (0,-2)$

$m = \max(cd \cap S)$

$U = ab(m)$
$L = S \setminus U$

$l = \min(cd \cap U)$
$k = \max(ld \cap L)$
$j = \min(ld \cap L)$

$x \in kl$

Fig. 2. Construction of a point generating a bounded component

1. To affinically transform a given simple closed curve to put it into a fixed rectangle.
2. To prove the Jordan curve theorem for such a curve, with following steps:
   (a) to construct a point supposed to be an element of a bounded component of the complement of the curve;
   (b) to prove that the component containing the point is really bounded;
   (c) to prove that the given curve is the border of any component of the complement of the curve;
   (d) to prove that every bounded component of the complement of the curve equals to the one containing the point.
3. The last result can be then obtain from the fact, that every homeomorphic image of a Jordan curve satisfies the Jordan property, as well.

Up to now we did not say any words why the Brouwer fixed point theorem and the Jordan curve theorem are connected. The place, where the first is used is Lemma 5.6 of [5], which proof goes as follows: if we assume on the contrary, that a given simple closed curve J is not a border of a given component, we can construct an arc being a part of J including the border. Then we can construct a disk centered at some point p belonging to a bounded component of the complement of J and including J. Next, using the Tietze extension theorem, formalized in [17], we can construct a particular map from the disk to disk with no fix points, what contradicts with the Brouwer fixed point theorem. Again, all details can be found in [5] and [16].

As we can see, the lemma refers to a couple of theorems with famous names. Figure 3 presents dependencies between them.

3.3 Formalization

Figure 4 presents some quantitative data about our formalization. Only main files have been taken into account with the calculations, that is files named: BORSUK_2 ([6]), BORSUK_6 ([7]), TOPALG_1 ([19]), TOPALG_2 ([12]), TOPALG_3 ([14]), TOPALG_4 ([13]), TOPALG_5 ([15]) in the section 'Algebraic topology'; BROUWER ([18]) in the section 'Brouwer'; URYSOHN1

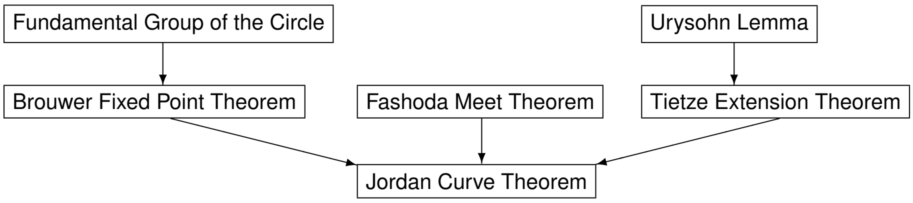


Fig. 3. Dependencies of famous lemmas used in the main proof

([1]), URYSOHN2 ([2]), URYSOHN3 ([3]) in the section 'Urysohn'; TIETZE ([17]) in the section 'Tietze'; JGRAPH_8 ([22]) in the section 'Fashoda'; and JORDAN ([16]) in the section 'Jordan'.

Theory	Articles	Lines	Bytes	Zip Bytes
Algebraic topology	7	16 364	560 856	105 172
Brouwer	1	1 900	65 679	13 319
Urysohn	3	7 132	248 030	37 599
Tietze	1	1 569	59 566	13 083
Fashoda	1	1 451	65 524	14 501
Jordan	1	6 801	229 257	41 077
total	14	35 217	1 228 912	224 751

Fig. 4. Numeric data

Acknowledgments

I would like to thank Professor Yatsuka Nakamura of Shinshu University for including me to the team working on the formalization of the Jordan curve theorem. I am also thankful to Professor Yasunari Shidama of Shinshu University for challenging me to formalize the Brouwer fixed point theorem. Especially, I am very grateful to Professors Nakamura and Shidama for funding me a scholarship and inviting me to Nagano to work on the project together. I am also thankful to Professor Andrzej Trybulec of University of Bialystok for his continual help and fruitful discussions during the formalization.

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