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Some Special Sequences of Points on the Plane

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Abstract - The article describes features of sequences and operations on sequences of points on the plane. All concepts are presented based on their definitions formalised in Mizar [2]. The article also characterises their implementation in the Jordan Curve Theorem (described in detail in [1]).

Keywords - formalized mathematics, Jordan Curve Theorem, sequences of points on the plane.

1. Preliminaries

As a plane we understand topological space R^2 with Euclid metric $\rho((x_1,y_1),(x_2,y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$, where (x_1,y_1) and (x_2,y_2) are points on the plane. A sequence of points on the plane is a sequence of which all elements belong to R^2. If $p = (x,y)$ is a point which belongs to R^2 then by p_x we will denote the abscissa of this point, i.e., x, whereas by p_y the ordinate of this point, i.e., y.

In the remainder of the article we assume that $\boldsymbol{a} = (a_1, a_2, ..., a_n)$ is a finite sequence of points on the plane unless stated otherwise.

Definition 1-1
Let $\boldsymbol{b}$, $\boldsymbol{c}$ be sequences of diverse variables in value and increasing created from all elements accordingly of set $\{(a_1)_x, (a_2)_x, ..., (a_n)_x\}$ and set $\{(a_1)_y, (a_2)_y, ..., (a_n)_y\}$. **Goboard** marked by sequence $\boldsymbol{a}$ is a matrix $GoB(\boldsymbol{a})$ of dimension $n \times n$, such that $[Gob(\boldsymbol{a})]_{ij} = (b_i,c_j)$.

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Definition 1-2

Let $p_1=(x_1,y_1)$ and $p_2=(x_2,y_2)$ be points of R^2. A segment marked by these points is called a set of points $\mathbf{L}(p_1,p_2) = \{(x,y): (x,y) = (1-\lambda)p_1 + \lambda p_2,\ 0 \le \lambda \le 1\}$.

Definition 1-3

Let i be a natural number such that $1 \le i \le n-1$. The segment marked by the i-th element of the sequence ***a*** we call a set of points $\mathbf{LSeg}(\mathbf{a},i) = L(a_i,a_{i+1})$. If $i > n-1$, it is assumed that $\mathbf{LSeg}(\mathbf{a},i) = \phi$.

Definition 1-4

A **polygonal path** marked by sequence ***a*** we call set $L\sim\mathbf{a}$ created out of all points of segments marked by this sequence, i.e., $L \sim \mathbf{a} = \bigcup_{1 \le i \le n-1} L(\mathbf{a}, i)$.

2. Features of Sequences

Definition 2-1

Sequence **a** is **special** (see [2]) if the abscissa or ordinate of neighbouring elements of this sequence are equal , i.e., $\forall i \in N\ 1 \le i \le n-1\ (a_i)_x = (a_{i+1})_x$ *lub* $(a_i)_y = (a_{i+1})_y$.

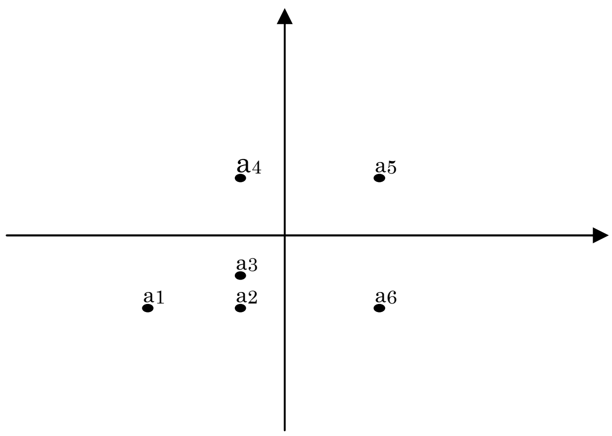


Figure 1. Special sequence

Definition 2-2

Sequence ***a*** is **unfolded** (see [2]) if the only common point of two segments is marked by a point which is the end of the first and the beginning of the second of these segments, i.e., $\forall i \in N\ 1 \le i \le n-2 \Rightarrow LSeg(\mathbf{a},i) \cap LSeg(\mathbf{a},i+1) = \{a_{i+1}\}$.

Definition 2-3

Sequence ***a*** is **circular** (see [5]) if the first and the last element of this sequence are equal.

Definition 2-4
Sequence ***a*** is a **simple non-closed curve** (see [2]) if every two segments marked by non-neighbouring elements of this sequence are separate, i.e.,
$\forall i,j \in N\ i+1<j \Rightarrow LSeg(a,i) \cap LSeg(a,j) = \phi$.

Definition 2-5
Sequence **a** is a **simple closed curve** (see [6]) if
$\forall i,j \in N\ i+1<j \wedge (i>1 \wedge j < n \vee j<n-1) \Rightarrow LSeg(a,i) \cap LSeg(a,j) = \phi$.

The difference between a simple non-closed curve and a simple closed curve is such that in the case of a simple closed curve it is not required that segments marked by the first and the last element of the sequence are disjoint.

Definition 2-6
Let there be a given matrix $[G]_{s \times t}$ with elements belonging to R^2. Sequence ***a*** is **sequence on the matrix** G if the following condition is fulfilled:
$a_k = g_{ij} \wedge a_{k+1} = g_{i',j'} \Rightarrow |i-i'|+|j-j'|=1$, where $1 \le k \le n-1$, $1 \le i, i' \le s$, $1 \le j, j' \le t$.

Definition 2-7
Sequence ***a*** is **standard** (see [6]) if it is a sequence on the matrix marked by this sequence, i.e., a sequence on the matrix *GoB(**a**)*.

A finite sequence of points on the plane which is of diverse value, unfolded, simple non-closed curve and special is called **S-Seq** (see [2]). Sequence **a** being S-Seq such that $a_1 = p_1$ i $a_n = p_2$, where p_1, p_2 are points belonging to R^2, is called a **special sequence joining** p_1,p_2 (see [10]).

Definition 2-8
Let $y_{N\text{-}most}$ be the highest value among ordinates of elements of a sequence ***a***, i.e., $y_{N\text{-}most} = max\{(a_1)_y,(a_2)_y, ...,(a_n)_y\}$. Let $A_{N\text{-}most}$ be a set of these elements of sequence ***a***, which has ordinate equal to $y_{N\text{-}most}$, whereas $p_{x\text{-}min}$ is an element of this set of smallest abscissa, i.e., $(p_{x\text{-}min})_x = min\ \{(a_i)_x\!: a_i \in A_{N\text{-}most}\}$. Sequence ***a*** is **clockwise oriented** (see [7]) if $(Rotate(\mathbf{a},p_{x\text{-}min}))_2 \in A_{N\text{-}most}$, (see def. 3-4), i.e., following element of the sequence ***a*** after the element $p_{x\text{-}min}$ has the same ordinate as element $p_{x\text{-}min}$.

3. Operations on Sequences

In the following, we distinguish three types of operations on sequences.

3.1 Modification of sequence in reference to one of its elements

Definition 3-1
*Min(**a**,x)* is the smallest number of the element of sequence ***a*** among numbers of these elements which are equal to *x*.

Example:
a *= (y,x,y,x,x,z) Min(**a**,x) = 2*

Let x be a point belonging to R^2.

Definition 3-2
Sequence ***a**:-x* (see [8]) is a subsequence of sequence ***a*** created in such a way that:
a**:-x* $= (x, a_{i+1}, ..., a_n)$, where $i = Min(\mathbf{a},x)$ that a_i is the first occurrence of the value x in sequence ***a.

Definition 3-3
Sequence ***a**-:x* (see [8]) is a subsequence of sequence ***a*** created in such a way that:
$a\text{-:}x = (a_1, a_2, ..., a_{i-1}, x)$, where $i = Min(a,x)$.

Definition 3-4
Sequence ***Rotate**(**a**,x)* (see [5]) is a sequence created from sequence **a**, in such a way that:
***Rotate**(**a**,x)* $= (x, a_{i+1}, a_{i+2}, ..., a_n, a_2, ..., a_{i-1}, x)$, where $i = Min(\mathbf{a},x)$.

Example:
a = (c,d,e,f,g,x,h,i) Rotate(a,x) = (x,h,i,d,e,f,g,x)

Definition 3-5
Let x be the value of an element of this sequence. Sequence ***a*** *-| x* (see [9]) is a sequence which is a cut of sequence ***a*** such as that ***a*** *-| x* $= (a_1, a_2, ..., a_{i-1})$, where $i = Min(\mathbf{a},x)$.

Definition 3-6
Let x be the value of an element of sequence ***a***. Sequence ***a*** |-- *x* (see [9]) is a subsequence of sequence ***a*** created from elements which occur after the first element of value *x*, i.e.,
a |-- $x = (a_{i+1}, ..., a_n)$, where $i = Min(\mathbf{a},x)$.

3.2 Other operations

Definition 3-7
Let ***a*** $= (a_1, a_2, ..., a_{n-1}, a_n)$. The reversed sequence (see [8]) is called sequence
$Rev(a) = (a_n, a_{n-1}, ..., a_2, a_1)$.

Definition 3-8
Sequence ***mid**(**a**,i,j)*, where $i,j \leq n$, (see [10]) is a subsequence of a sequence ***a*** created from elements out of numbers from *i* to *j* inclusive - in the following manner:

$$\mathbf{mid}(\mathbf{a},i,j) = \begin{cases} (a_i, a_{i+1}, ..., a_k, ..., a_{j-1}, a_j), \text{ where } & i \leq j \\ (a_j, a_{j-1}, ..., a_k, ..., a_{i+1}, a_i) \text{ otherwise.} & \end{cases}$$

Definition 3-9
Sequence *(i,j)-cut* ***a*** (see [11]) is a subsequence of a sequence ***a*** created from elements out of numbers from i to j inclusive - i.e., *(i,j)-cut* ***a*** = $(a_i, ..., a_j)$. If $i,j > n$ or $i > j$ it is assumed that sequence *(i,j)-cut* ***a*** is an empty set.

Definition 3-10
Let there be given a finite sequence ***a*** = $(a_1, a_2, ..., a_i, a_{i+1}, ..., a_n)$. Sequence ***Ins(a**,i,p)* (see [8]) is a sequence obtained by adding to sequence ***a*** a new element p between the i-th and $i+1$st elements, i.e., ***Ins(a**,n,p)* = $(a_1, a_2, ..., a_i, p, a_{i+1}, ..., a_n)$.

Definition 3-11
Let ***b*** = $(b_1, b_2, ..., b_m)$ be a finite sequence. Sequence ***a^'b*** = $(a_1, a_2, ..., a_n, b_2, ..., b_m)$ (see [11]).

3.3 Cutting sequences with respect to a point

Definition 3-12
Let p be a point belonging to R^2 so $p \in L$~***a***. ***Index**(p,**a**)* (see [10]) is the smallest number of the element of the sequence ***a*** among numbers of elements of the sequence ***a*** marking segments to which point p belongs (see Fig. 2).

Let p be a point belonging to R^2 such that *Index(p,a) = i*.

Definition 3-13
Sequence ***L_Cut(a**,p)* (see [10]) is a sequence defined in the following manner:

$$\textbf{\textit{L\_Cut}}(\textbf{\textit{a}},p) = \begin{cases} (p, a_{i+1}..., a_n) \text{ if } p \neq a_{i+1} \\ (p, a_{i+2}..., a_n) \text{ jeśli } p = a_{i+1} \end{cases}$$

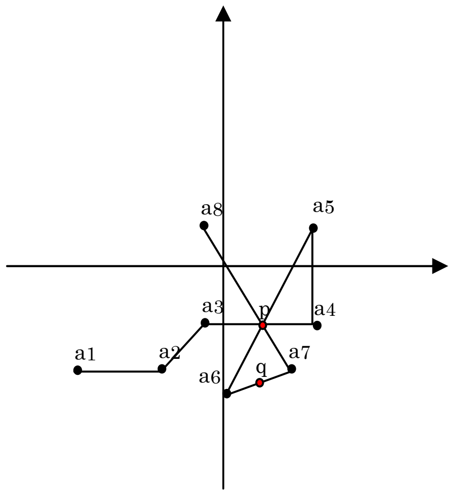
Definition 3-14
***R_Cut(a**,p)* (see [10]) is a sequence defined in the following manner:

$$\textbf{\textit{R\_Cut}}(\textbf{\textit{a}},p) = \begin{cases} (a_1, a_2, ..., a_i, p) \text{ if } p \neq a_1 \\ (p) \text{ if } p = a_1. \end{cases}$$

Definition 3-15
Let q be a point belonging to R^2 so ***Index**(q,**a**)* = j. ***B_Cut(a**,p,q)* (see [10]) is a sequence defined in the following manner:

$$\textbf{\textit{B\_Cut}}(\textbf{\textit{a}},p,q) = \begin{cases} R\_Cut(L\_Cut(a,p),q), \text{ if } i < j \text{ or } i = j \text{ and point } p \text{ is before point } q \\ Rev\ (R\_Cut(L\_Cut(a,q),p)) \text{ otherwise;} \end{cases}$$

Example



$Index(p, \mathbf{a}) = 3$

$L\_Cut(\mathbf{a}, p) = (p, a_4, a_5, \ldots, a_8)$

$R\_Cut(\mathbf{a}, p) = (a_1, a_2, a_3, p)$

$B\_Cut(\mathbf{a}, p, q) = (p, a_4, a_5, a_6, q)$

Figure 2. Operation on sequences.

4. Implementation

Polygonal paths marked by sequences of points on a plane are implemented for the approximations of curves. On these approximations the proof of the Jordan Curve Theorem is based, in which, at first, this theorem had been proved for the case of polygonal path approximations, and then it is used to prove the theorem for the general case (details in [1]). The sequence marking the polygonal path that approximates a curve from the outside is called **Cage**, whereas from the inside - **Span**. Formal definitions of these two concepts are found in articles MML [3,4]. Sequences marking such polygonal paths are: clockwise oriented, standard, special, circular sequence, unfolded, simple closed curve, and sequences on the matrix marked by curve (see Fig. 3).

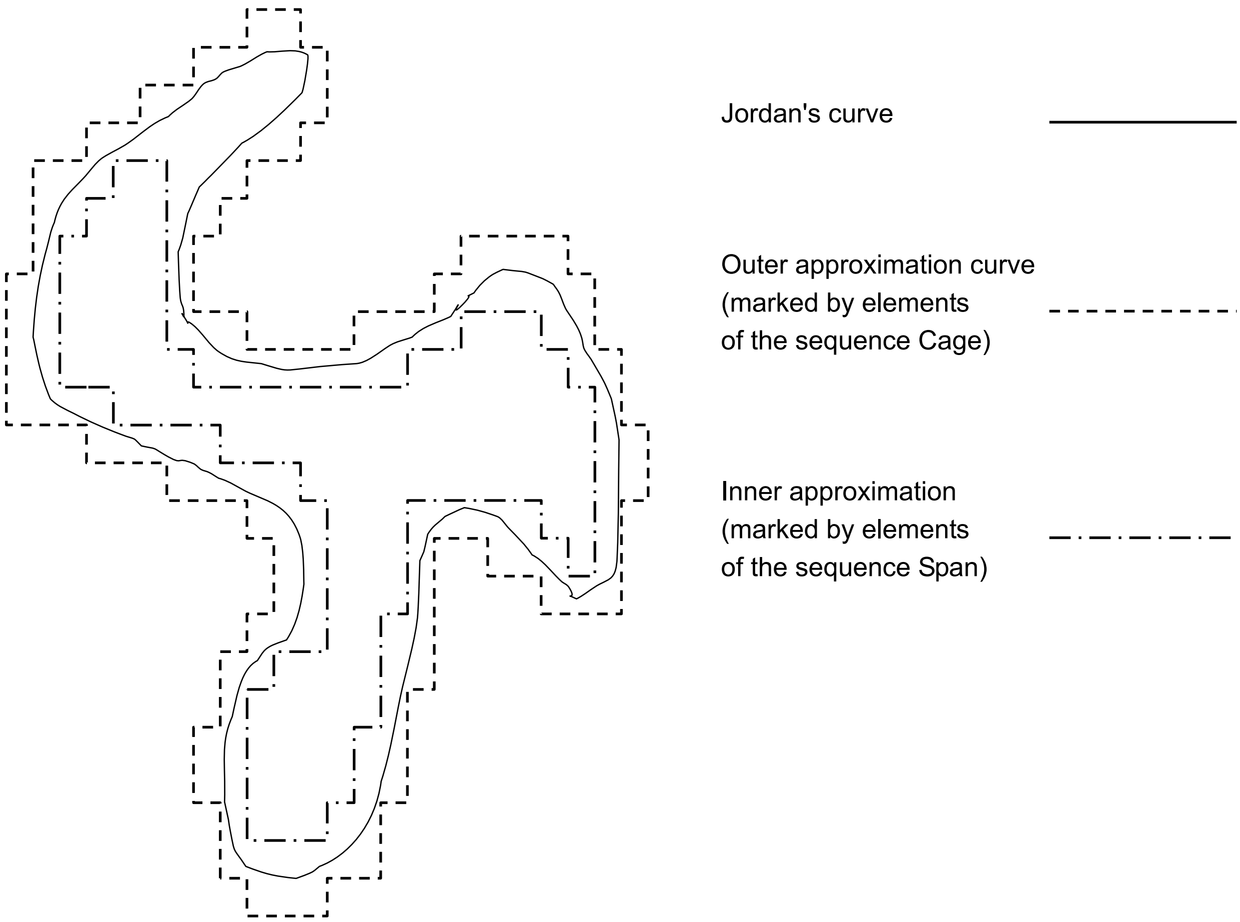


Figure 3. Polygonal paths marked by elements of sequence Cage and Span.

The enumerated kinds of sequences were also implemented in auxiliary theorems connected with the proof of the Jordan Curve Theorem, among others, in [14,15]. Special sequences of points on the plane have been used in the proof [1] of the Jordan Curve Theorem. Finally, in Mizar, the Jordan Curve Theorem was formalized in [12]. The original Jordan's Proof is explained in [13].

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