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Mizar Formalization of L^p Space

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Abstract - This paper shows codification of normed vector space of functions based on integrable functions over a measure space.

1 Introduction

Formalization of theory of function space, normed space, and Hilbert space of real sequences l^2 were developed and appeared in Mizar library in [1], [2], [3] and [4] respectively. We have discussed a space which is formed by integrable partial functions in [11], namely the set of all integrable partial functions has a linear space structure and becomes a real linear space by identifying two functions which are almost everywhere equal. Function space discussed in the article is based on a real valued partial function defined over non-empty set X. It is convenient for considering a function defined over various subsets when a measure is introduced in X. One often considers partial functions over a subset Y such that $X \neq Y$ and $\mu(X) = \mu(Y)$ (where μ is a measure of X). A space of partial functions can be enhanced to various spaces such as a normed space or a Banach space. We owed to the results of a measure theory and integral formalized in [7], [8],[9] and [10].

2 Algebraic Structure of Partial Functions

One can introduce a linear space structure by defining addition between functions and defining multiplication of a scalar to a function. It is straightforward to define addition and scalar multiplication to a set of partial functions by defining them point-wise substitution. The formalization is done by the following manner:

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Cited from Properties of Number-Valued Functions [12].
Definition
  let f1,f2 be complex-valued Function;
func f1 + f2 -> Function means
:: VALUED_1:def 1
  dom it = dom f1 /\ dom f2 & for c
  being set st c in dom it holds it.c = f1.c + f2.c;
  Commutative;
end;

definition
  let f be complex-valued Function, r be complex number;
  func r (#) f -> Function means
:: VALUED_1:def 5

  dom it = dom f & for c being set st c
  in dom it holds it.c = r * f.c;
end;
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The facts above show the set of partial functions from X to Real (denoted by PFuncs(X, REAL) has an algebraic structure. It is called by "RLSStruct", namely it has a monoid structure with re-

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spect to addition and has action of scalar multiplication of a real number on PFuncs(X, REAL). The structure above is formalized below:

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definition
  let A;
  func RLSp_PFunctA -> non empty RLSStruct equals
:: LPSPACE1:def 7
  RLSStruct(#PFuncs(A,REAL),
    RealPFuncZero A, addpfunc A, multrealpfunc A#);
end;
```

PFuncs(X, REAL) does not form an additive group because the neutral element cannot be defined or the inverse element cannot be uniquely defined. This cause is mainly due to freedom of choice of a partial function's domain. In order to make PFuncs(X, REAL) be a group, we need to introduce the following definitions or assumptions to additional constraints to the freedom of domain of functions:

1. X to be a measurable set,
2. a measurable function,
3. an integrable function,
4. two functions are almost everywhere equal ,
5. a function is almost everywhere defined on X.

3 Linear Space of Partial Function

Let X, S, M be a non empty set, a sigma field of subsets of X and a sigma measure, respectively. Throughout the rest of this paper, a partial function is defined over the measure space (X,S,M). The following definitions are to be recalled to construct function spaces.

1. (LPSPACE1: Def. 10)
   let f, g be elements of PFuncs(X, REAL). if f and g are measurable and $\exists N \in S$ st$M(N) = 0$ and $f|X\backslash N = g|X\backslash N$ then we say *f and g are almost everywhere equal* and denote as $f a.e. = gM$
   a.$e.$ = turns to be an equivalence relation on PFuncs(X, REAL) $\times$ PFuncs(X, REAL).
   The equivalence class includes f is denoted by a.e-eq-class(f, M)
2. let f be an element of PFuncs(X, REAL). f is called as a defined almost everywhere on X iff $\exists N \in S$ st $M(N) = 0$ and $dom f = X\backslash N$
3. (MESFUNC1: Def. 17)
   let f be an element of PFuncs(X, REAL). f is measurable on A iff
   $\forall r \in R, A \cap Lessdom(f, r) \in S.$ here $Lessdom(f, r) = \{x \in X | f(x) \leq r\}$
4. (MESFUNC5: Def. 16)
   Let f be an element of PFuncs(X, ExtREAL). The functor $\int f dM$ yielding an element of REAL is defined as follows:
   $$\int f dM = \int^{+} max_{+}(f) dM - \int^{+} max_{-}(f) dM$$
5. (MESFUNC5: Def. 17) We say that f is integrable on M if and only if:
   There exists an element A of S such that A = dom f and f is measurable on A and
   $$\int^{+} max_{+}(f) dM < +\infty \; and \int^{+} max_{-}(f) dM < +\infty$$

It is possible to construct a linear space from all the sets of partial functions which are defined almost everywhere on X, say a.e.Funcs(X, REAL). One can obtain linear space structure into a.e.Funcs(X, REAL) by taking the quotient of the equivalence relation of a.e.= . The same way is applied to the all the set of integrable functions and that would be much of interest.

4 L^p Space of Partial Function

let p be a positive real number and f be an element of PFuncs(X, REAL). f is said to be an element of L^p iff
f is defined as almost everywhere defined over X, is a measurable function and $|f|^p$ is integrable. The set of all the pth integrable function is denoted by Lp_Functions(M, p). For the case of $p = 1$, L1_Functions Mis used as in [11] instead of Lp_Functions(M, 1).

```
definition
  let X be non empty set, S be SigmaField of X, M be sigma_Measure of S,
  p be positive Real ;
  func Lp_Functions (M,p)  -> non empty Subset of RLSp_PFunct X
  equals
  :defLpF:
  { f  where f is PartFunc of X,REAL : ex Ef be Element of S st M.(Ef`)=0 &
  dom f = Ef & f is_measurable_on Ef & (abs f) to_power p is_integrable_on M };
end;
```

Lp_Functions(M, p) is not yet a linear space. However the residual set forms a linear space and internal and external operations are naturally defined.

Theorem 1 *A relation a.e.= with respect to M induces an equivalence relation on elements of* Lp_Functions(M, p) *and the equivalence classes have a linear space structure. Because one can determine an inverse of f of* Lp_Functions(M, p) *uniquely up to almost everywhere equal.* Lp_Functions(M, p)/ $\sim$ *is denoted by* Pre-Lp-Space(M, p).

Pre-Lp-Space (M,p) is presented by the Mizar system as stated below:

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definition
  let X be non empty set, S be SigmaField of X, M be sigma_Measure of S,
  k be  positive Real;
  func Pre-Lp-Space (M,k) -> strict Abelian add-associative right_zeroed
                       right_complementable RealLinearSpace-like
                       (non empty RLSStruct) means :VSPDef6X:
  the carrier of it = CosetSet (M,k) &
  the addF of it = addCoset (M,k) &
  0.it = zeroCoset (M,k) &
  the Mult of it = lmultCoset (M,k);
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5 Linear Normed Space of Partial Function

A norm is defined in Pre-Lp-Space(M, p) by taking an integral of a point of the space. The definition is coded as the following:

Definition 1 *let x be a point of* Lp_Functions(M, p). *A norm of the space* Pre-Lp-Space(M, p) *is defined by* Lp-Norm(M, k). Lp-Norm(M, k)(x) $= (\int |\ f\ |^p\ dM)^{1/p}$ *(where f is an element of the class of x).* Lp-Norm(M, k)(x) *can be expressed by $\|x\|_k$ in a mathematical context.*

Lp-Norm(M, k) satisfies the norm conditions, however it is required some technique to prove transitivity of the norm.

Theorem 2 *(Hölder's inequality)*
Let $m, n \in \mathbb{R}$ such that $1 \leq m < \infty$ and $\frac{1}{m} + \frac{1}{n} = 1$ and if $f \in$ Lp_ Functions(M, m) *and $g \in$* Lp_ Functions(M, n) *then $fg \in$* Lp_ Functions(M, 1) *and*
$\|f \cdot g\|_1 \quad \leq \quad \|f\|_m \cdot \|g\|_n$ *hold.*

(Codified Hölder's inequality.)

```
for X,S,M for f,g
for m,n be positive Real st 1/m +1/n =1 &
f in Lp_Functions (M,m) & g in Lp_Functions (M,n) holds
f(#)g in L1_Functions M & f(#)g is_integrable_on M

theorem Th001:
for X,S,M for f,g
for m,n be positive Real st 1/m +1/n =1 &
f in Lp_Functions (M,m) & g in Lp_Functions (M,n) holds
ex r1 be Real st r1 = Integral(M,(abs f) to_power m) &
ex r2 be Real st r2 = Integral(M,(abs g) to_power n) &
Integral(M,abs( f(#)g) ) <= r1 to_power (1/m) * r2 to_power (1/n)
```

Theorem 3 *(Minkowski's inequality)*
Suppose $1 \leq p < \infty$ *and if* $f \in \mathrm{Lp\_Functions(M, p)}$
and $g \in \mathrm{Lp\_Functions(M, p)}$.
Then $f + g \in \mathrm{Lp\_Functions(M, p)}$ *and* $\|f + g\|_p \leq \|f\|_p + \|g\|_p$ *hold.*

```
 (Codified Minkowski's inequality):
::Minkowski
theorem Th002X:
for X,S,M for f,g
for m be positive Real
for r1,r2,r3 be Element of REAL st
 1 <= m &
 f in Lp_Functions(M,m) & g in Lp_Functions(M,m) &
 r1 = Integral(M,(abs f) to_power m) &
 r2 = Integral(M,(abs g) to_power m) &
 r3 = Integral(M,(abs (f+g)) to_power m) holds
 r3 to_power (1/m) <= r1 to_power (1/m) + r2 to_power (1/m)
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We introduce a norm structure into Pre-Lp-Space(M,k) and call it Lp-Space(M,k). Due to the above theorems, it is verified that Lp-Space(M,k) is a real norm space. The space Lp-Space(M,k) is complete with respect to a metric induced from its norm. The completeness is defined by a Cauchy sequence of point of the space is convergent.

Lemma 1 *Let X be a real norm space, Sq be sequence of X, Sq0 be a point of X, R1 be a real sequence, and N be increasing sequence of* $\mathbb{N}$*. Suppose that Sq is a Cauchy sequence with respect to the norm of X and that for any* $i \in \mathbb{N}$ $R1(i) = \|Sq0 - Sq.(N(i))\|$ *and R1 is convergent and* $\lim_{x\to\infty} R1(x) = 0$*. Then Sq is convergent and* $\lim_{i\to\infty} Sq(i) = Sq0$ *and* $\|Sq(i) - Sq0\|$ *is convergent and* $\lim_{i\to\infty} \|Sq(i) - Sq0\| = 0$ *holds.*

Lemma 2 *: Let X and Sq be the same as the above lemma. Let Sq0 be a point of X. Suppose* $\|Sq - Sq0\|$ *is convergent and* $\lim_{i\to\infty} \|Sq(i) - Sq0\| = 0$ *then Sq is convergent and* $\lim_{i\to\infty} Sq(i) = Sq0$

Lemma 3 *: Let X and Sq be the same as the above lemma. Suppose Sq is Cauchy sequence by the norm of X then there exists an increasing sequence N of* $\mathbb{N}$ *satisfies* $\|Sq(j) - Sq(N(i))\| < 2^{-i}$ *for any two elements of* $\mathbb{N}$ *such that* $j \geq N(i)$ *holds.*

Lemma 4 *:* $(HOLDER1 : 10)$ *Let* $p \in \mathbb{N} and p > 0$ *and a,ap are real sequences. Suppose a is convergent,* $a_i \geq 0$ *and* $ap_i = a_i^p$*. Then ap is convergent and* $\lim_{i\to\infty} ap_i = \lim_{i\to\infty} a_i^p$ *holds.*

Theorem 4 *Monotone Convergence Theorem appeared in MESFUNC9:52*
Let E be an non-empty set, $\{f_i\}$ be a sequence of functions such that f_i is measurable, nonnegative and monotone increasing. Then the following holds.

$$\lim_{i\to\infty} f_i = f$$

$$\int_E \lim_{i\to\infty} f_i dM = \lim_{i\to\infty} \int_E f_i dM$$

Theorem 5 *: Let $k \in \mathbb{R}$ and $1 \leq k < \infty$. Let Sq be sequence of* $\text{Lp-Space}(M, k)$.
Then Sq is a Cauchy sequence w.r.t the norm of $\text{Lp-Space}(\text{M}, \text{k})$ *implies that Sq is convergent to an element of* $\text{Lp-Space}(\text{M}, \text{k})$.

Sketch of Proof Assume that $1 \leq k < \infty$ and $\{Sq_n\}$ is a Cauchy sequence in Lp-Space (M,k). There is a subsequence $\{Sq_{Ni}\}$ for some renumbering $N_1 < N_2 < \cdots < N_i$, such that

$$\|Sq(N_{i+1}) - Sq(N_i)\| < 2^{-i} \qquad (i = 1, 2, 3, \cdots) \tag{1}$$

by Lemma 3. We can take a sequence $\{F1_i\}$ such that $F1_i \in$ Lp_Functions(M,p) and $F1_i \in Sq(N_i)$. We put

$$G_i = |F1_0| + \sum_{j=1}^{i} |F1_{j+1} - F1_j|. \tag{2}$$

$G_i \in$ LpFunctions(M,p) holds for $i \in \mathbb{N}$. G_i is non-negative and $G_i \leq G_j$ for $i \leq j$.
The common domain of G_i is denoted by E_0. Namely

$$E_0 = \bigcap_{i=1}^{\infty} domG_i$$

and its complement $\bar{E_0}$ is measure zero. $G_i(x)$ is point-wise convergent on E_0, then we put

$$\lim_{i\to\infty} G_i = G$$

Now consider the following integral of (2). We obtain

$$\Big(\int_{E_0} (G_i)^p dM\Big)^{1/k} = \left(\int_{E_0} \Big(|F1_0| + \sum_{j=1}^{i} |F1_{j+1} - F1_j|\Big)^k dM\right)^{1/k}$$

$$\leq \Big(\int_{E_0} |F1_0|^k dM\Big)^{1/k} + \sum_{j=1}^{i} \Big(\int_{E_0} |F1_{j+1} - F1_j|^k dM\Big)^{1/k} \text{ by Theorem 3}$$

$$\leq \|Sq(N_1)\| + \sum_{j=1}^{i} \|Sq(N_{j+1} - Sq(N_j)\| \leq \|Sq(N_1)\| + 1 \textit{ by } (1).$$

Now apply Theorem 4 to $I(n) = \int_{E_0} G_n^p dM$.

$$\int_{E_0} G^k dM = \int_{E_0} \lim_{i\to\infty} G_i^k dM = \lim_{i\to\infty} \int_{E_0} G_i^k dM$$

$$= \lim_{i\to\infty} \|G_i\|^k \leq (\|Sq(N_1)\| + 1)^k$$

$$\leq \left(\Big(\int_{E_0} |F1_0|^k dM\Big)^{1/k} + 1\right)^k < +\infty$$

Therefore $G \in$ Lp_Functions(M,p). Since $F1_i|E_0$ is point-wise convergent, we name F is $\lim_{i\to\infty} F1_i|E_0$.

Evaluate $\int_{E_0} |F|^k dM$ then we know $F \in$ Lp_Functions (M,p). By translating property of $F1_i$ to $\{Sq_n\}$, we can conclude $\{Sq_n\}$ is convergent. □

A normed space is Banach space when it is complete. For the sake of the previous theorem, the next theorem holds.

Theorem 6 *: Let $k \in \mathbb{R}$ and $1 \leq k < \infty$.* Lp-Space(M, k) *is Banach space.*

6 Conclusion

As we have seen formal proofs regarding definitions and properties of L^p space which is rather an abstract object, a naive object namely a partial function has played an important role to formalize L^p space from the beginning. Our result would support our approach to formalize function space by using partial functions. It is remained to formalize the case when a function is essentially bounded namely the case of L^∞.

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